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A Lattice Boltzmann Equation for Waves

Yan Guangwu

State Key Laboratory of Nonlinear Mechanics (LNM), Institute of Mechanics, Chinese Academy of Sciences, Beijing 100080, People's Republic of China; [1]*Department of Mathematics, Jilin University, Changchun 130023, People's Republic of China*

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We propose a lattice Boltzmann model for the wave equation. Using a lattice Boltzmann equation and the Chapman–Enskog expansion, we get 1D and 2D wave equations with truncation error of order two. The numerical tests show the method can be used to simulate the wave motions. © 2000 Academic Press

Key Words: lattice Boltzmann method; wave equation; multiscale technique.

1. INTRODUCTION

In recent years, the lattice Boltzmann method (LBM) has attracted attention as an alternative numerical scheme for simulation of fluid flows [1–3, 10]. Unlike traditional numerical methods which solve equations for macroscopic variables, LBM is based on the mesoscopic kinetic equation for particle distribution function. The fundamental idea of the LBM is to construct a simplified kinetic model that incorporates the essential physics of microscopic or mesoscopic processes and where the macroscopic variables obey the desired macroscopic equations. In the general case, time, space, and velocity are discrete on one lattice. Noting this we choose the equilibrium distribution function to fit the same requirements which can be obtained with the multiscale technique and the Chapman–Enskog expansion. Recently, there have been some studies about model equations using the lattice Boltzmann method [4, 5, 9]. All of the models have a common characteristic: the macroscopic equations have term $\frac{\partial u}{\partial t}$ and convention term $\frac{\partial F(u)}{\partial \mathbf{x}}$, and the macroscopic quantity u is conservative. In other words, the macroscopic equation has the same form $\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial \mathbf{x}}$. However, the wave equation has another form

$$\frac{\partial^2 u}{\partial t^2} = C_s^2 \nabla^2 u + \psi(u),$$

[1] Present address.

where $\psi(u)$ is source term. The equation above has $\partial^2 u/\partial t^2$, but does not have the convection term.

It is known that the second-order wave equation can be transformed into a system of two first-order equations; thus, consequently, we need two types of particle distribution functions. This is a multi-composite lattice Boltzmann model, where at least a 10-bit model needs to be used. In this paper, we propose a new distribution function f_α for the macroscopic quantity $\frac{\partial u}{\partial t}$. We assume f_α satisfies the conservation condition. If we choose $\frac{\partial u}{\partial t}$ as a conservation quantity, the model has a 5-bit only. We can see the physical meaning in the wave equation: the quantity $\frac{\partial u}{\partial t}$ has an equilibrium distribution function in the mesoscopic scale from the view of our model, but the quantity u does not. The distribution function f_α is a change ratio of the number of particles with time, in the mesoscopic scale, not in the number of particles.

In Ref. [6], Chen and co-workers used three steps of lattice gas automata (LGA) to simulate a linear wave equation. They introduced a delay for time in LGA; the results are good enough. In the present paper, we also use a lattice model. Our model has three advantages over the LGA: (1) We use an exact solution of equilibrium distribution functions to determine the truncation errors of the scheme; the numerical results are quantity. (2) We do not need an ensemble average to get the macroscopic quantity; thus the statistical errors disappear. (3) We use the standard lattice Boltzmann equation that is simpler than the second-order differential type equation in LGA. The numerical results can show the mentioned advantages. In addition in this paper we use

$$u^{t+\Delta t} = u^t + \Delta t \frac{\partial u}{\partial t}$$

to find u of the next time step.

It is known that the second-order wave equation can be transformed into a system of two first-order equations. However, in a general system of two first-order equations, this is not easy to handle by standard LBM. Therefore, an alternative method for the lattice Boltzmann model of the second-order wave equation needs to be developed.

2. LATTICE BOLTZMANN MODEL

2.1. The definition of macroscopic quantity. Consider a 1D or 2D model; we make discrete the velocity of particles into b directions and a lattice with unit spacing is used where each node has b nearest neighbors connected by b links. The distribution function f_α is the probability of finding the quantity $\frac{\partial u}{\partial t}$ at time t, node $\mathbf{x}$, with velocity $\mathbf{e}_\alpha$, here $\alpha = 0, 1, \ldots, b$ ($\alpha = 0$ is the rest particle). The particle velocity is $\mathbf{e}_\alpha = \{0, c, -c\}$, $b = 2$ for a one-dimensional lattice; $\mathbf{e}_\alpha = \{(0, 0), (c, 0), (0, c), (-c, 0), (0, -c)\}$, $b = 4$ for a two-dimensional lattice. The macroscopic quantity $\frac{\partial u}{\partial t}$ is defined by

$$\frac{\partial u(\mathbf{x}, t)}{\partial t} = \sum_\alpha f_\alpha(\mathbf{x}, t); \tag{1}$$

the conservation condition is

$$\sum_\alpha f_\alpha^{eq}(\mathbf{x}, t) = \frac{\partial u(\mathbf{x}, t)}{\partial t}. \tag{2}$$

The distribution function satisfies the lattice Boltzmann equations (LBE)

$$f_\alpha(\mathbf{x}+\mathbf{e}_\alpha, t+1) - f_\alpha(\mathbf{x}, t) = \Omega_\alpha + \Omega'_\alpha, \tag{3}$$

where $\Omega_\alpha = -\frac{1}{\tau}[f_\alpha(\mathbf{x}, t) - f_\alpha^{eq}(\mathbf{x}, t)]$, f_α^{eq} is the equilibrium distribution function at time t, $\mathbf{x}$; Ω'_α is the additional term; and τ is the single relaxation time factor.

2.2. A series of lattice Boltzmann equations in different time scales. Using ε as the time step unit $\Delta t = \Delta x/c$ in the physical unit, ε can play the role of the Knudsen number [8]. The lattice Boltzmann equation (3) in physical units is

$$f_\alpha(\mathbf{x}+\varepsilon\mathbf{e}_\alpha, t+\varepsilon) - f_\alpha(\mathbf{x}, t) = \Omega_\alpha + \Omega'_\alpha. \tag{4}$$

The use of Taylor expansion gives

$$\begin{aligned} &f_\alpha(\mathbf{x}+\varepsilon\mathbf{e}_\alpha, t+\varepsilon) - f_\alpha(\mathbf{x}, t) \\ &\quad = \varepsilon\left[\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right] f_\alpha + \frac{\varepsilon^2}{2}\left[\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right]^2 f_\alpha + \frac{\varepsilon^3}{6}\left[\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}\right]^3 f_\alpha + O(\varepsilon^4). \end{aligned} \tag{5}$$

Next, the Chapman–Enskog expansion [7] is applied to f_α under the assumption that the mean free path is of the same order of ε. Expanding f_α around $f_\alpha^{(0)}$

$$f_\alpha = f_\alpha^{(0)} + \varepsilon f_\alpha^{(1)} + \varepsilon^2 f_\alpha^{(2)} + \varepsilon^3 f_\alpha^{(3)}, \ldots, \tag{6}$$

where $f_\alpha^{(0)}$ is f_α^{eq}.

To discuss changes in different time scales, introduce $t_0, \ldots, t_3$ as

$$t_0 = t, \qquad t_1 = \varepsilon t, \qquad t_2 = \varepsilon^2 t, \qquad t_3 = \varepsilon^3 t \tag{7}$$

and let

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + \varepsilon^2\frac{\partial}{\partial t_2} + \varepsilon^3\frac{\partial}{\partial t_3} + O(\varepsilon^4). \tag{8}$$

The equations to order of ε is

$$\frac{\partial f_\alpha^{(0)}}{\partial t_0} + e_{\alpha j}\frac{\partial f_\alpha^{(0)}}{\partial x_j} = -\frac{1}{\tau} f_\alpha^{(1)}. \tag{9}$$

Assume $\Omega'_\alpha = \varepsilon^2\phi(u)$. The equation to order of ε^2 is

$$\frac{\partial f_\alpha^{(0)}}{\partial t_1} - \tau\left(1 - \frac{1}{2\tau}\right)\left(\frac{\partial}{\partial t_0} + e_{\alpha j}\frac{\partial}{\partial x_j}\right)^2 f_\alpha^{(0)} = -\frac{1}{\tau} f_\alpha^{(2)} + \phi(u). \tag{10}$$

The equation to order of ε^3 is

$$\begin{aligned} &\frac{\partial f_\alpha^{(0)}}{\partial t_2} + (1-2\tau)\left(\frac{\partial}{\partial t_0} + e_{\alpha j}\frac{\partial}{\partial x_j}\right)\frac{\partial f_\alpha^{(0)}}{\partial t_1} + \left(\tau^2 - \tau + \frac{1}{6}\right)\left(\frac{\partial}{\partial t_0} + e_{\alpha j}\frac{\partial}{\partial x_j}\right)^3 f_\alpha^{(0)} \\ &\quad = -\frac{1}{\tau} f_\alpha^{(3)} + (-\tau)\left(\frac{\partial}{\partial t_0} + e_{\alpha j}\frac{\partial}{\partial x_j}\right)\phi(u). \end{aligned} \tag{11}$$

Equations (9)–(11) are a so-called series of lattice Boltzmann equations in different time scales [9].

2.3. The wave equation. Taking (9) + (10) × ε and summing about α, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = C_s^2 \frac{\partial^2 u}{\partial x_i \partial x_i} + \psi(u) + O(\varepsilon^2) \tag{12}$$

if

$$\sum_\alpha f_\alpha^{(0)} e_{\alpha i} = 0, \tag{13}$$

$$\sum_\alpha f_\alpha^{(0)} e_{\alpha i} e_{\alpha j} = \lambda u \delta_{ij}, \tag{14}$$

where $\lambda = C_s^2/\varepsilon(\tau - 1/2)$, $\psi(u) = \varepsilon(b+1)\phi(u)$. Equation (12) has the truncation error

$$R = O(\varepsilon^2). \tag{15}$$

To find the structure of the truncation error, we take (9) + (10) × ε + (11) × ε^2 and get

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= \lambda\varepsilon\left(\tau - \frac{1}{2}\right)\frac{\partial^2 u}{\partial x_i \partial x_i} + \varepsilon\phi(u)(b+1) \\ &\quad - 2\varepsilon^2\left(\tau^2 - \tau + \frac{1}{6}\right)\lambda\nabla^2\frac{\partial u}{\partial t} - \tau\varepsilon^2(b+1)\frac{\partial \phi}{\partial t} + O(\varepsilon^3).\end{aligned} \tag{16}$$

From Eq. (16), we get the truncation error as

$$R = -2\varepsilon^2\left(\tau^2 - \tau + \frac{1}{6}\right)\lambda\nabla^2\frac{\partial u}{\partial t} - \tau\varepsilon^2(b+1)\frac{\partial \phi}{\partial t} + O(\varepsilon^3). \tag{17}$$

The truncation error R contains a dissipation term and an unsteady source term. Therefore, on the second-level (ε^2), the term $\frac{\partial u}{\partial t}$ is dissipative with time.

We can easily get the equilibrium distribution function $f_\alpha^{(0)}$ from Eqs. (2), (13), and (14). The expression follows

$$f_0^{(0)} = \frac{\partial u}{\partial t} - \frac{\lambda u D}{c^2} \tag{18}$$

$$f_\alpha^{(0)} = \frac{\lambda u D}{bc^2}, \qquad \alpha = 1, \ldots, b, \tag{19}$$

where D is the dimensional number.

Equations (18) and (19) are a homogenous solution of the system. In the standard LBM, the square lattices with $b = 8$ (called an 8-bit) model is used generally. We can use the $b = 4$, $b = 6$, or $b = 8$ model for the 2D problem. As a simple model, we use $b = 4$ for the 2D problem to complete simulation in the numerical experiments of Section 3.

3. NUMERICAL EXAMPLE

To test the effect of this method, we choose three numerical experiments for the one-dimensional model and one numerical experiment for the two-dimensional model.

We select the lattice size M, the mesoscopic speed $c = |e_\alpha|$, the step of x as Δx, the speed of waves C_s, and the single relaxation time factor τ as parameters. The source terms $\psi(u)$ in the wave equation are a known condition. Thus, the length of the computing region is $l = M\Delta x$; the Knudsen number $\varepsilon = \Delta x/c$. The parameter λ in Eqs. (18) and (19) is given by $\lambda = C_s^2/\varepsilon(\tau - 1/2)$. The additional term Ω' in Eq. (3) is given by

$$\Omega' = \frac{\varepsilon\psi}{b+1}.$$

The initial conditions of distribution functions are given by Eqs. (18) and (19) from the macroscopic quantity u at time $t = t_0$. Starting from an initial f_α, the macroscopic quantity $\frac{\partial u}{\partial t}$ and u can be obtained using definitions. For each time step, the updating of the distribution function can be given by using the lattice Boltzmann equation (3). The boundary conditions of f_α are given by Eqs. (18) and (19) from the macroscopic quantity u on boundaries.

Test I,

$$u_{tt} = C_s^2 u_{xx} + g_0, \qquad 0 < x < \infty, t > 0$$
$$u(x, 0) = 0, \qquad u_t(x, 0) = 0,$$

and $u(0, t) = 0$, $u_x(x, t) \to 0$ when $x \to \infty$. The exact solution is

$$u(x, t) = \frac{g_0}{2}\left[t^2 - \left(t - \frac{x}{c}\right)^2 H\left(t - \frac{x}{c}\right)\right], \qquad t > 0, x > 0,$$

where $H(\xi)$ is the Heaviside function.

The results of the lattice Boltzmann simulation and theoretical solution are shown in Fig. 1. The parameters are lattice size $M = 100$, $\Delta x = 0.01$, $c = 5.0$, $\tau = 1.2$, $C_s = 0.1$, $g_0 = 0.1$, $\varepsilon = \Delta t = \Delta x/c$.

Test II,

$$u_{tt} = C_s^2 u_{xx}, \qquad 0 < x < 1, t > 0$$
$$u(x, 0) = 0, \qquad u_t(x, 0) = 0,$$
$$u(0, t) = 0, \qquad u_x(1, t) = g_0.$$

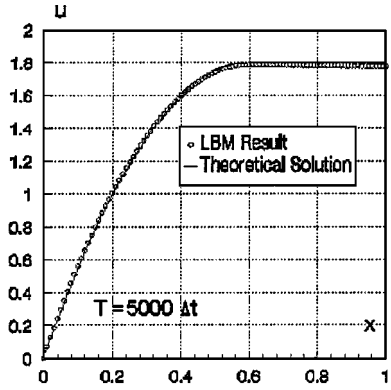

FIG. 1. Comparisons between numerical simulation (circle) and theoretical results (line) of Test I. Parameters: lattice size $M = 100$, $\Delta x = 0.01$, $c = 5.0$, $\tau = 1.2$, $C_s = 0.1$, $g_0 = 0.1$, $\varepsilon = \Delta t = \Delta x/c$.

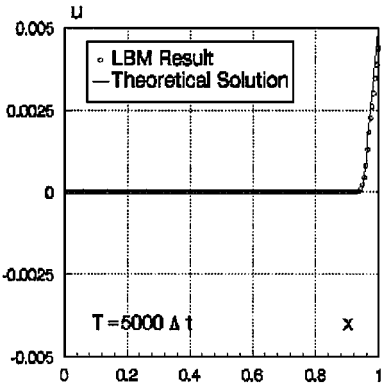

FIG. 2. Comparisons between numerical simulation (circle) and theoretical results (line) of Test II. Parameters: lattice size $M = 2000$, $\Delta x = 0.0005$, $c = 5.0$, $\tau = 1.2$, $C_s = 0.01$, $g_0 = 0.1$, $\varepsilon = \Delta t = \Delta x/c$.

The exact solution is

$$u(x,t) = C_s g_0 \left[\left(t - \frac{1-x}{C_s}\right) H\left(t - \frac{1-x}{C_s}\right) - \left(t - \frac{1+x}{C_s}\right) H\left(t - \frac{1+x}{C_s}\right) \right.$$
$$\left. - \left(t - \frac{3-x}{C_s}\right) H\left(t - \frac{3-x}{C_s}\right) + \left(t - \frac{3+x}{C_s}\right) H\left(t - \frac{3+x}{C_s}\right) \right], \quad 0 < t < \frac{4}{C_s}.$$

The results of the lattice Boltzmann simulation and theoretical solution are shown in Fig. 2. The parameters are lattice size $M = 2000$, $\Delta x = 0.0005$, $c = 5.0$, $\tau = 1.2$, $C_s = 0.01$, $g_0 = 0.1$, $\varepsilon = \Delta t = \Delta x/c$.

Test III,

$$u_{tt} = C_s^2 u_{xx}, \qquad t > 0, -\infty < x < \infty$$
$$u(x,0) = \frac{0.2}{1+9x^2}, \qquad u_t(x,0) = 0.$$

The exact solution is

$$u(x,t) = \frac{0.1}{1+9(x-C_s t)^2} + \frac{0.1}{1+9(x+C_s t)^2}. \tag{20}$$

We plot the wave motion at four moments in Figs. 3a–3d. The parameters are lattice size $M = 1000$, $\Delta x = 0.01$, $c = 3.0$, $\tau = 1.2$, $C_s = 0.1$, $\varepsilon = \Delta t = \Delta x/c$. We find that the single wave packet evolves into a right-traveling wave packet and a left-traveling wave packet and the shapes are preserved at all times. We also plot the errors in Fig. 3e at time $T = 3000\Delta t$. We use the function $e(x,t) = |(u(x,t) - u^*(x,t))/u^*(x,t)|$ as the errors, where $u^*(x,t)$ is the exact solution in Eq. (20). Comparing similar examples in Ref. [6], we find three aspects differing from the LGA model: (1) We use an exact solution of equilibrium distribution functions Eqs. (18), (19) to determine the truncation errors of the scheme; our numerical results are quantity. (2) We do not need an ensemble average to get the macroscopic quantity; thus the statistical errors disappear. (3) We use a standard lattice Boltzmann equation that is simpler than the second-order differential type equation in LGA; especially, our model can be used to simulate the wave equation with source term $\psi(u)$.

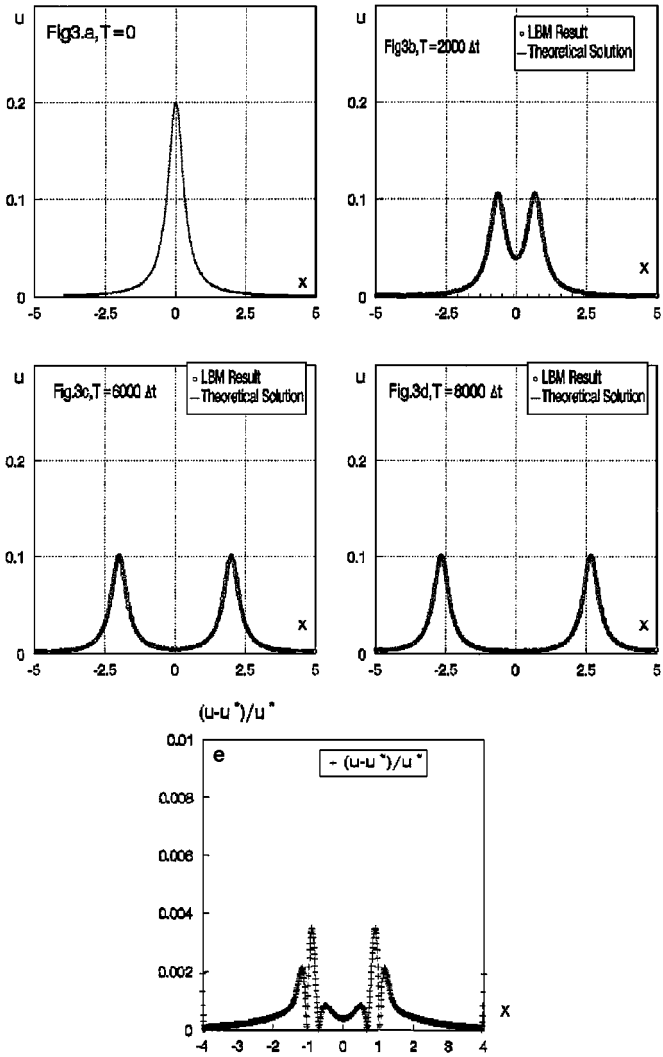

FIG. 3. (a)–(d) Comparisons between numerical simulation (circle) and theoretical results (line) of Test III. Parameters: lattice size $M = 1000$, $\Delta x = 0.01$, $c = 3.0$, $\tau = 1.2$, $C_s = 0.1$, $\varepsilon = \Delta t = \Delta x/c$. (e) The errors curve verses position x of Test III at time $T = 3000\Delta t$. Parameters: lattice size $M = 1000$, $\Delta x = 0.01$, $c = 3.0$, $\tau = 1.2$, $C_s = 0.1$, $\varepsilon = \Delta t = \Delta x/c$.

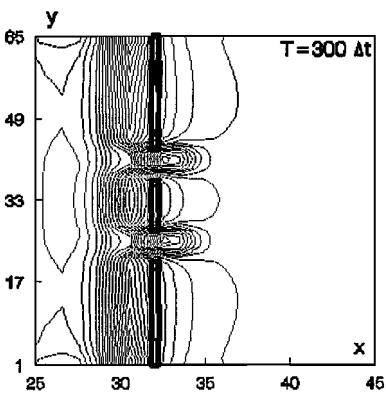

FIG. 4. Contours of quantity u at $T = 300\Delta t$. Lattice size 64×64; the slits position cell numbers $m_1 = 32$, $n_1 = 25$, $m_2 = 32$, $n_2 = 41$; the slits width $\Delta l = 6$, $\Delta x = 1/64$, $c = 3.0$, $\delta = 5.0$, $\tau = 1.2$, $C_s = 0.1$, $g_0 = 0.1$, $\varepsilon = \Delta t = \Delta x/c$.

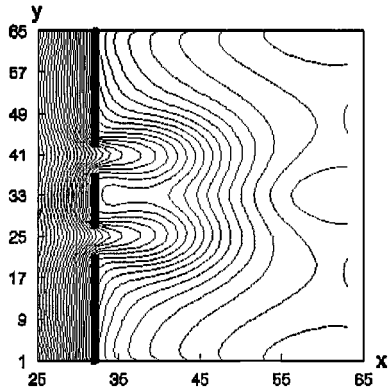

FIG. 5. Contours of quantity u at $T = 1000\Delta t$. Lattice size 64×64; the slits position cell numbers $m_1 = 32$, $n_1 = 25$, $m_2 = 32$, $n_2 = 41$; the slits width $\Delta l = 6$, $\Delta x = 1/64$, $c = 3.0$, $\delta = 2.0$, $\tau = 1.2$, $C_s = 0.1$, $g_0 = 0.1$, $\varepsilon = \Delta t = \Delta x/c$.

Test IV. As a 2D experiment, we give the results of double-slit at a given instant [6]; see Figs. 4, 5. We put a plane wave as the initial condition,

$$u(x, y, 0) = g_0 \cos(\delta x).$$

The boundary conditions on the left, right, top, under, and holes are grad $u = 0$. On two sides of the wall, we use $u = 0$. At 32 cells away from the left boundary we put a wall with two holes each with width Δl, at n_1, n_2 cells away from under the boundary, so that the wave can go through the holes in order to go from the left region into the right region (we put this as empty initially). This results in the interference patterns.

4. CONCLUSION

In this paper, we have presented a lattice Boltzmann model for the linear wave equation. Some key points are as follows.

(1) The lattice Boltzmann method is available for the linear wave equation. Because the linear wave equation does not have the convention term, its lattice Boltzmann model has a feature of $\sum_\alpha f_\alpha^{eq} e_\alpha = 0$. This is different from the standard LBM in Refs. [1–3].

(2) We used three time scales for recovery of the macroscopic equation. If we want to know a more detailed structure in the coefficient of dissipation and dispersion, more higher order moments of the f_α^{eq} should be used.

(3) The difference between this model and other lattice Boltzmann models is that $\frac{\partial u}{\partial t}$ but not u is the conservation quantity. Test results support the theory.

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